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Trentelman, H.L.; Napp Avelli, D.

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On the regular implementability of nD systems

H.L. Trentelman*, D. Napp Avelli

Mathematics Institute, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands

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Abstract

In this short paper we study the problem of finding necessary and sufficient conditions for regular implementability by partial interconnection for nD system behaviors. In [M.N. Belur, H.L. Trentelman, Stabilization, pole placement and regular implementability, IEEE Trans. Automat. Control 47(5) (2002) 735–744.] such conditions were obtained in the context of 1D systems. In the present paper we show that the conditions obtained in [M.N. Belur, H.L. Trentelman, Stabilization, pole placement and regular implementability, IEEE Trans. Automat. Control 47(5) (2002) 735–744.] are no longer valid in general in the nD context. We also show that under additional assumptions, the conditions still remain relevant. We also reinvestigate the conditions for regular implementability by partial interconnection in terms of the canonical controller that were obtained in [P. Rocha, Canonical controllers and regular implementation of nD behaviors, Proceedings of the 16th IFAC World Congress, Prague, Czech Republic, 2005.]. Using the geometry of the underlying modules we generalize a result on regular implementability from the 1D to the nD case. Finally, we study how, in the 1D context, the conditions from [M.N. Belur, H.L. Trentelman, Stabilization, pole placement and regular implementability, IEEE Trans. Automat. Control 47(5) (2002) 735–744; P. Rocha, Canonical controllers and regular implementation of nD behaviors, Proceedings of the 16th IFAC World Congress, Prague, Czech Republic, 2005.] are connected.

Keywords: Behavioral control; nD systems; Regular implementability; Canonical controller

1. Introduction

A fundamental question in control is to characterize, for a given plant to be controlled, the achievable limits of performance. In the behavioral approach to control this problem has been formalized as the problem of characterizing all behaviors that are implementable with respect to the given plant behavior. Originally the problem was mainly studied for 1D systems, see for example [14,2,1,7], but also generalizations to more general classes of system behaviors, including nD systems, have been investigated, see [12,10,4,15].

We will review the concept of implementability. Suppose we have a system behavior with two types of variables, the variable to be controlled w and the variable c through which the system can be interconnected to a controller behavior. This system behavior is called the *full plant behavior*, where *full* refers to the fact that we consider both types of variables w and c in specifying the behavior. To interconnect the full plant to a controller means requiring that the c trajectories in the full plant behavior are also elements of the controller behavior. The space of w trajectories in the interconnection of full plant and controller is called the *manifest controlled behavior*. A given ('desired') behavior is called *implementable by partial interconnection* (through c) if it can be obtained as manifest controlled behavior. A given behavior is called *regularly implementable* if it can be obtained as manifest controlled behavior using a controller behavior that does not impose restrictions on the control variables that were already present in the full plant behavior.

Given a 1D full plant behavior, in [14] for the first time a characterization was given of all implementable behaviors. Later, in [12], this result was generalized to more general system classes, including nD system behaviors. A characterization of all regularly implementable behaviors, in the 1D context, was established for the first time in [2], see also [1,13]. At approximately the same time, in the nD context, in [10] and also in [15] necessary and sufficient conditions were given for regular

* Corresponding author. Tel.:+31 50 3633998; fax: +31 50 3633800.
E-mail address: h.l.trentelman@math.rug.nl (H.L. Trentelman).

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implementability by *full interconnection*, the special case that the c variable and the w variable coincide. More recently, in [10] regular implementability by partial interconnection was investigated also in the nD context. In the present short paper, our aim is to reinvestigate the problem of regular implementability by partial interconnection for nD behaviors.

In this paper we denote the polynomial ring $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_n]$ of polynomials with real coefficients, in n indeterminates by $\mathscr{D}$. By $\mathscr{A}^q$ we denote the space $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q)$ of all infinitely often differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^q$. The results in this paper remain valid also for $\mathscr{A}^q = \mathfrak{D}'(\mathbb{R}^n, \mathbb{R}^q)$, the space of all $\mathbb{R}^q$-valued distributions on $\mathbb{R}^n$.

2. Implementability

In this section we review some concepts of nD behavioral systems. For a nice overview we refer to, for example, [6,10,15].

In the behavioral approach to nD systems, a *behavior* is a subset of the space $\mathbb{W}^{\mathbb{T}}$ consisting of all trajectories from $\mathbb{T}$, the *indexing set*, to $\mathbb{W}$, the *signal space*. Here, we consider systems with $\mathbb{T} = \mathbb{R}^n$ and $\mathbb{W} = \mathbb{R}^q$. We call $\mathfrak{B}$ a *linear differential nD behavior* or simply: *linear nD behavior* if it is the solution set of a system of linear, constant-coefficient partial differential equations, more precisely, if $\mathfrak{B}$ is the subset of $\mathscr{A}^q$ consisting of all solutions to

$$R\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) w = 0, \tag{1}$$

where R is a polynomial matrix in n indeterminates ξ_i, $i = 1, \ldots, n$, and $\mathrm{d}/\mathrm{d}\mathbf{x} = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. We call (1) a *kernel representation* of $\mathfrak{B}$ and we write $\mathfrak{B} = \ker(R)$. Obviously, any linear differential nD behavior $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$. Linear differential nD behaviors can have different representations as well. If $M \in \mathscr{D}^{q\times m}$ then the representation $\mathfrak{B} = \{w \in \mathscr{A}^q \mid \text{there exists } \ell \in \mathscr{A}^m \text{ s.t. } w = M(\mathrm{d}/\mathrm{d}\mathbf{x})\ell\}$, is called an *image representation* of $\mathfrak{B}$ and we write $\mathfrak{B} = \mathrm{im}(M)$.

It was shown in [5] that there is a one-to-one correspondence between linear differential nD behaviors and submodules of $\mathscr{D}^q$. This one-to-one correspondence is valid for the choices $\mathscr{A}^q = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q)$ or $\mathfrak{D}'(\mathbb{R}^n, \mathbb{R}^q)$, but not for arbitrary choice of signal space. With any linear differential nD behavior $\mathfrak{B} \subseteq \mathscr{A}^q$ we associate the submodule $\mathscr{M}(\mathfrak{B})$ of $\mathscr{D}^q$ defined by

$$\mathscr{M}(\mathfrak{B}) := \left\{ r \in \mathscr{D}^q \,\middle|\, r\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) w = 0 \text{ for all } w \in \mathfrak{B} \right\}.$$

Conversely, for any submodule $\mathscr{M}$ of $\mathscr{D}^q$ we have that

$$\mathfrak{B}(\mathscr{M}) := \left\{ w \in \mathscr{A}^q \,\middle|\, r\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\right) w = 0 \text{ for all } r \in \mathscr{M} \right\}$$

is a linear differential nD behavior. With this bijection, we have $\mathscr{M}(\mathfrak{B}_1 \cap \mathfrak{B}_2) = \mathscr{M}(\mathfrak{B}_1) + \mathscr{M}(\mathfrak{B}_2)$ and $\mathfrak{B}(\mathscr{M}_1 \cap \mathscr{M}_2) = \mathfrak{B}(\mathscr{M}_1) + \mathfrak{B}(\mathscr{M}_2)$. Again, these statements hold for our choice of signal space $\mathscr{A}^q$, but are not true in general, see e.g. [11]. If $\mathfrak{B} = \ker(R)$ then $\mathscr{M}(\mathfrak{B})$ is the submodule of $\mathscr{D}^q$ of all $\mathscr{D}$-linear combinations of the rows of R. This submodule is denoted by $\langle R \rangle$.

Given a $\mathscr{D}$-module $\mathscr{M}$, an element $m \in \mathscr{M}$ is called a *torsion element* if there exists $0 \neq d \in \mathscr{D}$ such that $dm = 0$. The set of torsion elements is a submodule of $\mathscr{M}$. If this submodule is the 0-module, then $\mathscr{M}$ is called *torsion-free*.

A behavior $\mathfrak{B}$ is called *regular* if the module $\mathscr{M}(\mathfrak{B})$ is *free*, equivalently if there exists a polynomial matrix R of full row rank such that $\mathfrak{B} = \ker(R)$. In contrast with the case $n = 1$, for $n \geqslant 2$ not all behaviors are regular.

A polynomial matrix $R \in \mathscr{D}^{g\times q}$ is called zero left prime (ZLP) if its gth order minors generate the polynomial ring $\mathscr{D}$ as an ideal. A linear nD behavior $\mathfrak{B}$ is called *strongly controllable* if there exists a ZLP R such that $\mathfrak{B} = \ker(R)$. A polynomial matrix $R \in \mathscr{D}^{g\times q}$ is called zero right prime if its transpose R^{T} is ZLP.

$R \in \mathscr{D}^{g\times q}$ is called a *minimal left annihilator* (MLA) of $M \in \mathscr{D}^{q\times m}$ if $\ker(R) = \mathrm{im}(M)$.

In this paper, if we consider the direct sum $\mathscr{A}^{q_1} \oplus \mathscr{A}^{q_2}$, then the submodules $\mathscr{A}^{q_1} \oplus \{0\}$ and $\{0\} \oplus \mathscr{A}^{q_2}$ will be denoted by $\mathscr{A}^{q_1}$ and $\mathscr{A}^{q_2}$, respectively. Likewise, $\mathscr{D}^{q_1}$ and $\mathscr{D}^{q_2}$ are considered as submodules of $\mathscr{D}^{q_1} \oplus \mathscr{D}^{q_2}$.

Let $\mathfrak{B} \subseteq \mathscr{A}^{q_1} \oplus \mathscr{A}^{q_2}$ be a linear nD behavior with system variable $w = (w_1, w_2)$, where $w_i \in \mathbb{R}^{q_i}$, $i = 1, 2$. Let pr_1 denote the projection of $\mathscr{A}^{q_1} \oplus \mathscr{A}^{q_2}$ onto $\mathscr{A}^{q_1}$. Then the subspace $\mathrm{pr}_1(\mathfrak{B}) \subseteq \mathscr{A}^{q_1}$ is again a linear nD behavior. Indeed, using the fundamental principle of Ehrenpreis and Palamodov it can be shown that if $\mathfrak{B} = \ker(R_1\ R_2)$, then a kernel representation of $\mathrm{pr}_1(\mathfrak{B})$ is constructed as follows: take a MLA F of R_2. Then $\mathrm{pr}_1(\mathfrak{B}) = \ker(FR_1)$, see [5, Corollary 2.38]. Taking into account the partition $w = (w_1, w_2)$, the module associated with $\mathfrak{B}$ is a submodule $\mathscr{M}$ of $\mathscr{D}^{q_1} \oplus \mathscr{D}^{q_2}$. Clearly, the module of $\mathrm{pr}_1(\mathfrak{B})$ equals $\mathscr{M} \cap \mathscr{D}^{q_1}$.

Assume now we have a linear differential nD behavior $\mathscr{P}_{\mathrm{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ with system variable (w, c), where w takes values in $\mathbb{R}^q$ and c in $\mathbb{R}^k$, to be interpreted as a plant to be controlled. Let $\mathfrak{C} \subseteq \mathscr{A}^k$ be an nD behavior with system variable c, called a controller. The interconnection of $\mathscr{P}_{\mathrm{full}}$ and $\mathfrak{C}$ through c is defined as the nD behavior $\mathscr{K}_{\mathrm{full}}(\mathfrak{C}) := \mathscr{P}_{\mathrm{full}} \cap (\mathscr{A}^q \oplus \mathfrak{C})$, called the *full controlled behavior*. The projection $\mathrm{pr}_1(\mathscr{K}_{\mathrm{full}}(\mathfrak{C}))$ onto $\mathscr{A}^q$ is called the *manifest controlled behavior*. In terms of the associated modules, the module of $\mathscr{P}_{\mathrm{full}}$ is a submodule $\mathscr{M}$ of $\mathscr{D}^q \oplus \mathscr{D}^k$ and the module of $\mathfrak{C}$ is a submodule $\mathscr{C}$ of $\mathscr{D}^k$. The module of the full controlled behavior is equal to $\mathscr{M} + \mathscr{C}$, while the module of the manifest controlled behavior equals $(\mathscr{M} + \mathscr{C}) \cap \mathscr{D}^q$.

The interconnection of $\mathscr{P}_{\mathrm{full}}$ and $\mathfrak{C}$ through c is called *regular* if their modules intersect trivially, i.e. $\mathscr{M} \cap \mathscr{C} = 0$. This can be interpreted as saying that, in a regular interconnection, the controller does not reimpose conditions that were already present in the plant.

Let $\mathscr{K} \subseteq \mathscr{A}^q$ be a linear nD behavior, to be interpreted as a 'desired' behavior. If $\mathfrak{C} \subseteq \mathscr{A}^k$ is such that $\mathscr{K} = \mathrm{pr}_1(\mathscr{K}_{\mathrm{full}}(\mathfrak{C}))$, then we say that $\mathfrak{C}$ implements $\mathscr{K}$ by partial interconnection (w.r.t. $\mathscr{P}_{\mathrm{full}}$). If $\mathscr{M} \subseteq \mathscr{D}^q \oplus \mathscr{D}^k$ is the module of $\mathscr{P}_{\mathrm{full}}$, $\mathscr{N}$ the module of $\mathscr{K}$, and $\mathscr{C}$ the module of $\mathfrak{C}$, then $\mathfrak{C}$ implements $\mathscr{K}$ by partial interconnection if and only if $(\mathscr{M} + \mathscr{C}) \cap \mathscr{D}^q = \mathscr{N}$. If, in addition, the interconnection is regular, equivalently $\mathscr{M} \cap \mathscr{C} = 0$, then we say that $\mathfrak{C}$ regularly implements $\mathscr{K}$.

We call $\mathscr{K} \subseteq \mathscr{A}^q$ implementable by partial interconnection if there exists $\mathfrak{C} \subseteq \mathscr{A}^k$ that implements $\mathscr{K}$. $\mathscr{K}$ is called regularly implementable by partial interconnection if there exists $\mathfrak{C}$ that regularly implements $\mathscr{K}$.

In addition to partial interconnection, we look at *full interconnection*. If in $\mathscr{P}_{\text{full}}$ w coincides with c, so if interconnection takes place through the to be controlled variable, then we speak about full interconnection. In that case it is more natural to consider the plant as a behavior $\mathscr{P}$ with one variable w through which also the interconnection takes place. The (full) interconnection with a controller $\mathfrak{C}$ is then defined as the intersection $\mathscr{P} \cap \mathfrak{C}$. The interconnection is regular if $\mathscr{M}(\mathscr{P}) \cap \mathscr{M}(\mathfrak{C}) = \{0\}$.

A given $\mathscr{K} \subseteq \mathscr{A}^q$ is implementable by full interconnection if there exists $\mathfrak{C} \subseteq \mathscr{A}^q$ such that $\mathscr{P} \cap \mathfrak{C} = \mathscr{K}$, and regularly implementable by full interconnection if this condition holds for some $\mathfrak{C}$ while the interconnection is regular. In terms of the corresponding modules, $\mathscr{K}$ is implementable if and only if there exists a submodule $\mathscr{C} \subseteq \mathscr{D}^q$ such that $\mathscr{M}(\mathscr{P}) + \mathscr{C} = \mathscr{M}(\mathscr{K})$. $\mathscr{K}$ is regularly implementable if and only if there exists a submodule $\mathscr{C} \subseteq \mathscr{D}^q$ such that $\mathscr{M}(\mathscr{P}) \oplus \mathscr{C} = \mathscr{M}(\mathscr{K})$, stated differently, $\mathscr{M}(\mathscr{P})$ is a direct summand of $\mathscr{M}(\mathscr{K})$. In the remarkable paper [15, Theorem 3.2] this condition was shown to be equivalent to the solvability of a linear polynomial matrix equation (see also [3]).

Necessary and sufficient conditions for implementability by partial interconnection for a given $\mathscr{K} \subseteq \mathscr{A}^q$ for the case $n = 1$ are given in [14]. In [12] it was shown that these conditions are also necessary and sufficient for more general classes of linear systems, including nD systems. To make this paper self-contained, we review these conditions here. For a given linear nD full plant behavior $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ we call $\text{pr}_1(\mathscr{P}_{\text{full}})$ *the manifest plant behavior*. The intersection $\mathscr{P}_{\text{full}} \cap \mathscr{A}^q$ is called the *hidden behavior*. An important role is played by the so-called *canonical controller* (see [12]). For a given $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ we define the canonical controller by $\mathfrak{C}_{\text{can}}(\mathscr{K}) := \text{pr}_2(\mathscr{P}_{\text{full}} \cap (\mathscr{K} \oplus \mathscr{A}^k))$. The following holds:

Proposition 1. *Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ be linear nD systems. $\mathscr{K} \subseteq \mathscr{A}^q$ is implementable by partial interconnection if and only if $\mathscr{P}_{\text{full}} \cap \mathscr{A}^q \subseteq \mathscr{K} \subseteq \text{pr}_1(\mathscr{P}_{\text{full}})$.*

Proof. ($\Leftarrow$) We first prove that

$$\mathscr{P}_{\text{full}} \cap (\mathscr{K} \oplus \mathscr{A}^k) = \mathscr{P}_{\text{full}} \cap (\mathscr{A}^q \oplus \mathfrak{C}_{\text{can}}(\mathscr{K})). \tag{2}$$

The inclusion '$\subseteq$' is immediate. To prove '$\supseteq$', let (w, c) be an element of the right-hand side of (2). Then $(w, c) \in \mathscr{P}_{\text{full}}$ and $(0, c) \in \mathfrak{C}_{\text{can}}(\mathscr{K})$. By definition of the canonical controller, there exists w' such that $(w', c) \in \mathscr{P}_{\text{full}} \cap (\mathscr{K} \oplus \mathscr{A}^k)$. Thus $(w, c) = (w - w', 0) + (w', c)$, which yields $(w - w', 0) \in \mathscr{P}_{\text{full}} \cap \mathscr{A}^q$. Now use that the hidden behavior is contained in $\mathscr{K}$ to deduce that $(w - w', 0) \in \mathscr{K} \oplus \mathscr{A}^k$. We conclude that $(w, c) \in \mathscr{K} \oplus \mathscr{A}^k$, so an element of the left-hand side of (2). Finally, using the inclusion $\mathscr{K} \subseteq \text{pr}_1(\mathscr{P}_{\text{full}})$, it is easily checked that $\text{pr}_1(\mathscr{P}_{\text{full}} \cap (\mathscr{K} \oplus \mathscr{A}^k)) = \mathscr{K}$. In view of (2) this implies that the canonical controller implements $\mathscr{K}$. ($\Rightarrow$). Let $\mathfrak{C}$ be such that $\text{pr}_1(\mathscr{P}_{\text{full}} \cap (\mathscr{A}^q \oplus \mathfrak{C})) = \mathscr{K}$. It is then straightforward to check $\mathscr{P}_{\text{full}} \cap \mathscr{A}^q \subseteq \mathscr{K} \subseteq \text{pr}_1(\mathscr{P}_{\text{full}})$. $\square$

Note that if $\mathscr{M} \subseteq \mathscr{D}^q \oplus \mathscr{D}^k$ is the module of $\mathscr{P}_{\text{full}}$, then $\mathscr{M} \cap \mathscr{D}^q$ is the module of $\text{pr}_1(\mathscr{P}_{\text{full}})$. The module of the hidden behavior is equal to $\text{pr}_1(\mathscr{M})$, where pr_1 is the projection of $\mathscr{D}^q \oplus \mathscr{D}^k$ onto $\mathscr{D}^q$. Denoting the module of $\mathscr{K}$ by $\mathscr{N}$, $\mathscr{K}$ is implementable by partial interconnection if and only if $\mathscr{M} \cap \mathscr{D}^q \subseteq \mathscr{N} \subseteq \text{pr}_1(\mathscr{M})$.

We now turn to conditions for *regular* implementability by partial interconnection. For the 1D case, in [2] the following proposition was proven:

Proposition 2. *Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ be linear 1D systems. Then $\mathscr{K}$ is regularly implementable by partial interconnection if and only if the following two conditions hold*:

1. *$\mathscr{K}$ is implementable by partial interconnection w.r.t. $\mathscr{P}_{\text{full}}$,*
2. *$\mathscr{K}$ is regularly implementable by full interconnection with respect to $\text{pr}_1(\mathscr{P}_{\text{full}})$.*

One aim of this paper is to study whether the above characterization of regular implementability also holds in the context of nD systems for $n \geqslant 2$. This will turn out to be not the case. However, under additional assumptions the above conditions will turn out to remain valid.

An alternative characterization of regular implementability by partial interconnection was given for the 1D case in [8], and for the general nD case in [9]. This characterization is in terms of the canonical controller:

Proposition 3. *Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ be linear nD systems. Then $\mathscr{K}$ is regularly implementable by partial interconnection w.r.t. $\mathscr{P}_{\text{full}}$ if and only if*

1. *$\mathscr{K}$ is implementable by partial interconnection w.r.t. $\mathscr{P}_{\text{full}}$,*
2. *$\mathfrak{C}_{\text{can}}(\mathscr{K})$ is regularly implementable by full interconnection w.r.t. $\text{pr}_2(\mathscr{P}_{\text{full}})$.*

A second aim of this paper is to re-investigate for nD behaviors the role of the canonical controller in the problem of (regular) implementability. We will do this by carefully analyzing the geometry of the underlying modules. This will enable us to derive some new results on implementability and regular implementability of nD systems. We will also investigate the connection between the respective conditions 2 appearing in the above propositions.

3. Does Proposition 1 hold for nD systems?

In this section we will show that in the nD context, neither the 'if' statement nor the 'only if' statement in Proposition 2 are valid in general. We will, however, provide additional assumptions under which the 'if' statement and the 'only if' statement do remain valid.

First, we will give a counterexample to the 'if' statement, more concretely, give two examples in which $\mathscr{K}$ is regularly implementable by full interconnection and implementable by partial interconnection, but not regularly implementable by partial interconnection. After discussing the examples, we will prove that the 'if' part does hold if we assume that $\mathscr{K}$ is regularly implementable by full interconnection *using a regular controller*, i.e. a controller whose module is free.

Example 4. Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^2 \oplus \mathscr{A}^2$ be represented by $R_1(\mathrm{d}/\mathrm{d}\mathbf{x})w + R_2(\mathrm{d}/\mathrm{d}\mathbf{x})c = 0$, with

$$R_1(\xi_1,\xi_2) := \begin{pmatrix} 1 & 0 \\ 0 & \xi_1 \\ 0 & \xi_2 \end{pmatrix}, \quad R_2(\xi_1,\xi_2) := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take $\mathscr{K} = \ker(R_1) = \mathscr{P}_{\text{full}} \cap \mathscr{A}^2$ (the hidden behavior). We compute $\mathrm{pr}_1(\mathscr{P}_{\text{full}}) = \ker(1\ 0)$. We see that $\mathscr{P}_{\text{full}} \cap \mathscr{A}^2 = \mathscr{K} \subseteq \mathrm{pr}_1(\mathscr{P}_{\text{full}})$ so $\mathscr{K}$ is implementable by partial interconnection. It is also regularly implementable by full interconnection with respect to $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$: as controller take $\mathfrak{C}_1 = \ker(C_1)$ with

$$C_1(\xi_1,\xi_2) = \begin{pmatrix} 0 & \xi_1 \\ 0 & \xi_2 \end{pmatrix}.$$

The full interconnection of $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ and $\mathfrak{C}_1$ is regular since $\langle (1\ 0)\rangle \cap \langle (0\ \xi_1), (0\ \xi_2)\rangle = 0$. We now show that $\mathscr{K}$ is *not* regularly implementable by partial interconnection. Let $\mathfrak{C} = \ker(C)$ be a controller that acts on c and that implements $\mathscr{K}$. We claim that necessarily $\mathfrak{C} = \{0\}$. Assume, on the contrary, there exists a trajectory $(w_1, w_2, c_1, c_2) \in \mathscr{K}_{\text{full}}(\mathfrak{C})$ with $(c_1, c_2) \neq (0, 0)$. Then we must have $\partial w_2/\partial x_1 = -c_1$ and $\partial w_2/\partial x_2 = -c_2$. This contradicts the fact that $(w_1, w_2) \in \mathscr{K}$, so w_2 must be constant. Thus, indeed, $\mathfrak{C} = \{0\}$. This, however, implies that C must be zero right prime, so in particular $\mathrm{rank}(C) = 2$ so C has full column rank. Thus we find that $\mathrm{rank}\begin{pmatrix} R_1 & R_2 \\ 0 & C \end{pmatrix} = \mathrm{rank}\begin{pmatrix} R_1 & 0 \\ 0 & C \end{pmatrix} = 4$, while $\mathrm{rank}(R_1\ R_2) = 3$. We conclude that the partial interconnection of $\mathscr{P}_{\text{full}}$ and $\mathscr{C}$ is not regular.

A second example is given below. The details are left to the reader.

Example 5. Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^4 \oplus \mathscr{A}^3$ be represented by $R_1(\mathrm{d}/\mathrm{d}\mathbf{x})w + R_2(\mathrm{d}/\mathrm{d}\mathbf{x})c = 0$, with

$$R_1(\xi_1,\xi_2,\xi_3) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\xi_3 & \xi_2 \\ 0 & \xi_3 & 0 & -\xi_1 \\ 0 & -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

$$R_2(\xi_1,\xi_2,\xi_3) := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Take $\mathscr{K} := \ker(R_1)$. As in the previous example it can be shown that $\mathscr{K}$ is implementable by partial interconnection, regularly implementable by full interconnection with respect to $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$, but not regularly implementable by partial interconnection.

We now prove that the 'if' part of Proposition 2 remains valid in the nD case under the additional assumption that $\mathscr{K}$ is regularly implementable by full interconnection with a regular controller, i.e. a controller that admits a full row rank kernel representation.

Theorem 6. *Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ be linear nD systems. If $\mathscr{K}$ is implementable by partial interconnection, and regularly implementable by full interconnection with respect to $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ by means of a regular controller, then it is regularly implementable by partial interconnection.*

Proof. Let $\mathscr{N} \subseteq \mathscr{D}^q$ be the module of $\mathscr{K}$ and $\mathscr{M} \subseteq \mathscr{D}^q \oplus \mathscr{D}^k$ the module of $\mathscr{P}_{\text{full}}$. Let N and $(R_1\ R_2)$ be polynomial matrices such that $\langle N\rangle = \mathscr{N}$ and $\langle (R_1\ R_2)\rangle = \mathscr{M}$. By assumption there exists a free module $\overline{\mathscr{C}} \subseteq \mathscr{D}^q$ such that $(\mathscr{M} \cap \mathscr{D}^q) \oplus \overline{\mathscr{C}} = \mathscr{N}$. Let $\overline{C}$ be a polynomial matrix with linearly independent rows such that $\langle \overline{C}\rangle = \overline{\mathscr{C}}$. There exists a polynomial matrix W such that $\overline{C} = WN$. Also, since $\mathscr{N} \subseteq \mathrm{pr}_1(\mathscr{M})$, there exists L such that $N = LR_1$. Define now $C := WLR_2$, and let $\mathscr{C}$ be the module generated by the rows of C. We claim that $\mathscr{M} \cap \mathscr{C} = \{0\}$. Indeed, let $m \in \mathscr{M} \cap \mathscr{C}$. Then there exist polynomial row vectors r and s such that

$$m = r(R_1\ R_2) = s(0\ WLR_2).$$

This implies

$$(r - sWL)(R_1\ R_2) = (-sWLR_1\ 0) = (-s\overline{C}\ 0) =: n.$$

The vector n thus defined is in $\mathscr{M} \cap \mathscr{D}^q \cap \overline{\mathscr{C}} = \{0\}$, so $s\overline{C} = 0$, which implies that $s = 0$. This yields $m = 0$.

Next, we prove that $\mathscr{M} + \mathscr{C} = \mathscr{M} + \overline{\mathscr{C}}$. Let $c \in \overline{\mathscr{C}}$. There is a polynomial row vector r such that $c = r(\overline{C}\ 0) = r(WN\ 0) = r(WLR_1\ 0) = r(WLR_1\ WLR_2) - r(0\ C)$, which is obviously in $\mathscr{M} + \mathscr{C}$. The converse is proven in the same way. Finally, since $\overline{\mathscr{C}} \subseteq \mathscr{D}^q$, we have $(\mathscr{M} + \overline{\mathscr{C}}) \cap \mathscr{D}^q = (\mathscr{M} \cap \mathscr{D}^q) + \overline{\mathscr{C}} = \mathscr{N}$. This proves that the controller $\mathfrak{C} = \ker(C)$ regularly implements $\mathscr{K}$ by partial interconnection. $\square$

Remark 7. As a consequence of the above theorem we obtain that if $\mathscr{K}$ is a *regular* behavior, implementable by partial interconnection, and regularly implementable by full interconnection w.r.t. $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$, then $\mathscr{K}$ is regularly implementable by partial interconnection. Indeed, if $\mathscr{N}$ is a free module and $\overline{\mathscr{C}}$ is a direct summand of $\mathscr{N}$, then by the theorem of Quillen and Suslin $\overline{\mathscr{C}}$ is a free module as well.

Example 8. As in the 1D case, in the general nD case regularity of $\mathscr{K}$ is, however, *not a necessary condition*. Take $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^2 \oplus \mathscr{A}^1$ with $R_1(\xi_1, \xi_2)$ as in Example 4, and

$$R_2(\xi_1,\xi_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Take $\mathscr{K} = \ker(R_1)$. $\mathscr{K}$ is a non-regular behavior. The manifest plant behavior equals $\mathrm{pr}_1(\mathscr{P}_{\text{full}}) = \ker(R)$, with

$$R(\xi_1,\xi_2) = \begin{pmatrix} 0 & \xi_1 \\ 0 & \xi_2 \end{pmatrix}.$$

It can be verified that $\mathscr{K}$ is regularly implemented by full interconnection w.r.t. $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ by the controller $\ker(1\ 0)$. Note that this controller is a regular behavior, so the condition of Theorem 6 is satisfied. $\mathscr{K}$ is regularly implemented by partial interconnection w.r.t. $\mathscr{P}_{\text{full}}$ by the controller $\{0\}$.

We now turn to the 'only if' condition of Proposition 2. In the general nD context also the 'only if' part of this proposition does not hold. A counterexample of a behavior $\mathscr{K}$ that is regularly implementable by partial interconnection, but not by full interconnection with respect to the manifest plant behavior was given in [9]. In this section we restate this counterexample for the continuous time case. We also prove a theorem stating that under additional assumptions on the full plant behavior the 'only if' part remains valid in the nD case.

Example 9. Consider $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^2 \oplus \mathscr{A}^1$ represented by $R_1(\mathrm{d}/\mathrm{d}\mathbf{x})w + R_2(\mathrm{d}/\mathrm{d}\mathbf{x})c = 0$, with

$$R_1(\xi_1, \xi_2) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2(\xi_1, \xi_2) := \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Take $\mathscr{K} = \{0\}$, which is regularly implemented by partial interconnection using the controller $c = 0$. We compute $\mathrm{pr}_1(\mathscr{P}_{\text{full}}) = \ker(R)$, with $R(\xi_1, \xi_2) := (\xi_2 - \xi_1)$. We claim that $\mathscr{K}$ is not regularly implementable by full interconnection w.r.t. $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$. Indeed, this would be equivalent to the statement that $\langle(\xi_1\ -\xi_1)\rangle$ is a direct summand of $\mathscr{D}^2$, which is clearly not the case.

Theorem 10. *Let $\mathscr{P}_{\text{full}} \subset \mathscr{A}^q \oplus \mathscr{A}^k$ be a linear nD behavior. Let $\mathscr{M}$ be the module of $\mathscr{P}_{\text{full}}$. Assume that $\mathscr{M} \cap \mathscr{D}^q$ is a direct summand of $\mathscr{M}$. Then for any linear nD behavior $\mathscr{K} \subseteq \mathscr{A}^q$ we have: if $\mathscr{K}$ is regularly implementable by partial interconnection then $\mathscr{K}$ is regularly implementable by full interconnection with respect to* $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$.

Proof. Let $\mathscr{N}$ be the module of $\mathscr{K}$. There exists a module $\mathscr{C} \subset \mathscr{D}^k$ such that $(\mathscr{M} + \mathscr{C}) \cap \mathscr{D}^q = \mathscr{N}$ and $\mathscr{M} \cap \mathscr{C} = \{0\}$. Let $\mathscr{M}'$ be such that $(\mathscr{M} \cap \mathscr{D}^q) \oplus \mathscr{M}' = \mathscr{M}$. Define a module $\overline{\mathscr{C}} \subseteq \mathscr{D}^q$ by

$$\overline{\mathscr{C}} := (\mathscr{M}' + \mathscr{C}) \cap \mathscr{D}^q.$$

We claim that $(\mathscr{M} \cap \mathscr{D}^q) \oplus \overline{\mathscr{C}} = \mathscr{N}$. Indeed, $(\mathscr{M} \cap \mathscr{D}^q) + \overline{\mathscr{C}} = [\underline{(\mathscr{M} \cap \mathscr{D}^q) + \mathscr{M}'} + \mathscr{C}] \cap \mathscr{D}^q = (\mathscr{M} + \mathscr{C}) \cap \mathscr{D}^q = \mathscr{N}$. If $m \in \mathscr{M} \cap \mathscr{D}^q \cap \overline{\mathscr{C}}$ then $m = m' + c$ with $m' \in \mathscr{M}'$ and $c \in \mathscr{C}$. Since $m' \in \mathscr{M}$ we have $c \in \mathscr{M} \cap \mathscr{C} = \{0\}$. This implies $m = m' \in \mathscr{M}'$, so $m \in (\mathscr{M} \cap \mathscr{D}^q) \cap \mathscr{M}' = \{0\}$. $\square$

Thus, under the condition that $\mathscr{M} \cap \mathscr{D}^q$ is a direct summand of $\mathscr{M}$, every $\mathscr{K}$ that is regularly implementable by partial interconnection is regularly implementable by full interconnection w.r.t. $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$. We will now investigate this direct summand condition.

Clearly in the 1D case the condition always holds. Indeed, if $\mathscr{M} = \langle(R_1\ R_2)\rangle$, then by unimodular premultiplication we can obtain

$$U(R_1\ R_2) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & 0 \end{pmatrix},$$

with R_{12} full row rank and $\mathscr{M} \cap \mathscr{D}^q = \langle(R_{21}\ 0)\rangle$. Thus, $\langle(R_{11}\ R_{12})\rangle$ is a direct summand. In the general nD case the condition does not hold in general. We do have the following:

Proposition 11. *Let $\mathscr{M} = \langle(R_1\ R_2)\rangle$ be a submodule of $\mathscr{D}^q \oplus \mathscr{D}^k$. Let F be a MLA of R_2. Then $\mathscr{M} \cap \mathscr{D}^q$ is a direct summand of $\mathscr{M}$ if and only if the equation*

$$(FY - I)FR_1 = 0 \tag{3}$$

has a polynomial solution Y. In that case, a direct summand of $\mathscr{M} \cap \mathscr{D}^q$ is generated by the rows of the polynomial matrix $((YF - I)R_1\ R_2)$.

Proof. A proof of this proposition can be given by applying [15, Theorem 3.2]. $\square$

Note that (3) has a solution if F is ZLP, equivalently, $\mathrm{im}(R_2)$ is strongly controllable. Thus a sufficient condition for the direct summand condition is that $\mathrm{im}(R_2)$ is strongly controllable.

Example 12. Again consider the system $\mathscr{P}_{\text{full}}$ of Example 4. Obviously, $\mathrm{im}(R_2)$ is strongly controllable, so the direct summand condition of Theorem 10 is satisfied. Let $\mathscr{K} = \ker(K)$, with

$$K(\xi_1, \xi_2) := \begin{pmatrix} 1 & 0 \\ 0 & \xi_1^2 + \xi_2^2 \end{pmatrix}.$$

$\mathscr{K}$ is regularly implemented by partial interconnnection by the controller $\ker(C)$ with $C(\xi_1, \xi_2) = (\xi_1\ \xi_2)$. It is regularly implemented by full interconnection w.r.t. $\mathscr{P} = \ker(1\ 0)$ by the controller $\ker(0\ \xi_1^2 + \xi_2^2)$.

4. The canonical controller and regular implementability

In this section we will study the role of the canonical controller in implementability. First, we will restate and sharpen some of the results on implementability and regular implementability that already appeared in [9]. We will, however, rather work with the underlying submodules of $\mathscr{D}^q \oplus \mathscr{D}^k$ than with the behaviors themselves. The section will close with some new results on regular implementability by partial interconnection.

Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ be linear nD behaviors, with corresponding modules $\mathscr{M} \subseteq \mathscr{D}^q \oplus \mathscr{D}^k$ and $\mathscr{N} \subseteq \mathscr{D}^q$, respectively. Recall from Section 2 the definition of the canonical controller associated with $\mathscr{K}$. Obviously, the module of $\mathfrak{C}_{\text{can}}(\mathscr{K})$ is equal to $(\mathscr{M} + \mathscr{N}) \cap \mathscr{D}^k$. It is easily verified that this module equals $\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$, where

$$\tilde{\mathscr{M}}\ (\mathscr{N}) := \{(m_1, m_2) \in \mathscr{M} \mid (m_1, 0) \in \mathscr{N}\}. \tag{4}$$

Lemma 13. *Assume that $\mathscr{M} \cap \mathscr{D}^q \subseteq \mathscr{N} \subseteq \mathrm{pr}_1(\mathscr{M})$. Let $\mathscr{C} \subseteq \mathscr{D}^k$ be a submodule. Then the following hold*:

1. $(\mathscr{M} + \mathscr{C}) \cap \mathscr{D}^q \subseteq \mathscr{N} \Leftrightarrow \mathscr{C} \cap \mathrm{pr}_2(\mathscr{M}) \subseteq \mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$,
2. $(\mathscr{M} + \mathscr{C}) \cap \mathscr{D}^q = \mathscr{N} \Leftrightarrow (\mathscr{C} \cap \mathrm{pr}_2(\mathscr{M})) + (\mathscr{M} \cap \mathscr{D}^k) = \mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$,

3. $((\mathscr{M}+\mathscr{C})\cap\mathscr{D}^q=\mathscr{N}$ *and* $\mathscr{M}\cap\mathscr{C}=\{0\})\Leftrightarrow(\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M}))\oplus(\mathscr{M}\cap\mathscr{D}^k)=\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$,
4. $\mathscr{C}+(\mathscr{M}\cap\mathscr{D}^k)=\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))\Rightarrow(\mathscr{M}+\mathscr{C})\cap\mathscr{D}^q=\mathscr{N}$,
5. $\mathscr{C}\oplus(\mathscr{M}\cap\mathscr{D}^k)=\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))\Rightarrow((\mathscr{M}+\mathscr{C})\cap\mathscr{D}^q=\mathscr{N}$ *and* $\mathscr{M}\cap\mathscr{C}=\{0\})$.

Proof. (1) ($\Rightarrow$) Let $(0,c)\in\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M})$. There exists m_1 such that $(m_1,c)\in\mathscr{M}$. Then $(m_1,0)=(m_1,c)-(0,c)\in(\mathscr{M}+\mathscr{C})\cap\mathscr{D}^q\subseteq\mathscr{N}$, so $(0,c)\in\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$.

($\Leftarrow$) Let $(m_1,m_2)+(0,c)\in\mathscr{D}^q$ with $(m_1,m_2)\in\mathscr{M}$ and $(0,c)\in\mathscr{C}$. Then $m_2=-c$. Now, $(0,c)=(0,-m_2)\in\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M})\subseteq\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. Thus, there exists n_1 such that $(n_1,0)\in\mathscr{N}$ and $(n_1,-m_2)\in\mathscr{M}$. This yields $(m_1,m_2)+(n_1,-m_2)=(m_1+n_1,0)\in\mathscr{M}\cap\mathscr{D}^q\subseteq\mathscr{N}$, so $(m_1,0)\in\mathscr{N}$. Conclude that $(m_1,m_2)+(0,c)=(m_1,0)\in\mathscr{N}$.

(2) ($\Rightarrow$) Let $(0,m_2)\in\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. There exists m_1 such that $(m_1,0)\in\mathscr{N}$ and $(m_1,m_2)\in\mathscr{M}$. Also, there exists m_2' such that $(m_1,m_2')\in\mathscr{M}$ and $(0,-m_2')\in\mathscr{C}$. Hence $(0,m_2)=(m_1,m_2)-(m_1,m_2')+(0,m_2')$. The sum of the first two terms on the right is in $\mathscr{M}\cap\mathscr{D}^k$, the third term on the right is in $\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M})$.

($\Leftarrow$) Let $(m_1,0)\in\mathscr{N}$. Since $\mathscr{N}\subseteq\mathrm{pr}_1(\mathscr{M})$ there exists m_2 such that $(m_1,m_2)\in\mathscr{M}$, so $(m_1,m_2)\in\tilde{\mathscr{M}}(\mathscr{N})$, and $(0,m_2)\in\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. Consequently, $(0,m_2)=(0,c)+(0,m_2')$ with $(0,c)\in\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M})$ and $(0,m_2')\in\mathscr{M}\cap\mathscr{D}^k$. We conclude that $(m_1,0)=(m_1,m_2)-(0,c)-(0,m_2')\in(\mathscr{M}+\mathscr{C})\cap\mathscr{D}^q$.

(3) ($\Rightarrow$) If $\mathscr{M}\cap\mathscr{C}=\{0\}$, then obviously the terms on the right also have a zero intersection.

($\Leftarrow$) Let $(m_1,m_2)\in\mathscr{M}\cap\mathscr{C}$. Then of course $(m_1,m_2)=(0,m_2)$ must be in $\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M})$, but also in $\mathscr{M}\cap\mathscr{D}^k$, so must be equal to 0.

(4) We claim that $(\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M}))+(\mathscr{M}\cap\mathscr{D}^k)=\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. The implication then follows from (2) above. Indeed, let $(0,m_2)\in\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. Then there exists m_1 such that $(m_1,0)\in\mathscr{N}$ and $(m_1,m_2)\in\mathscr{M}$. By assumption, $(0,m_2)$ can be written as $(0,m_2)=(0,c)+(0,m_2')$ with $(0,c)\in\mathscr{C}$ and $(0,m_2')\in\mathscr{M}\cap\mathscr{D}^k$. Clearly, $(0,m_2)\in\mathrm{pr}_2(\mathscr{M})$ and $(0,m_2')\in\mathrm{pr}_2(\mathscr{M})$, so $(0,c)\in\mathrm{pr}_2(\mathscr{M})$.

(5) The only thing left to prove here is that $\mathscr{M}\cap\mathscr{C}=\{0\}$, which is obvious.

From this lemma we immediately reobtain most of the results on implementability and regular implementability of nD behaviors from [9]:

Corollary 14. *Assume that* $\mathscr{A}^q\cap\mathscr{P}_{\text{full}}\subseteq\mathscr{K}\subseteq\mathrm{pr}_1(\mathscr{P}_{\text{full}})$. *Let* $\mathfrak{C}\subseteq\mathscr{A}^k$ *be a linear nD behavior. Then we have*

1. $\mathscr{K}\subseteq\mathrm{pr}_1(\mathscr{K}_{\text{full}}(\mathfrak{C}))\Leftrightarrow\mathfrak{C}_{\text{can}}(\mathscr{K})\subseteq\mathfrak{C}+(\mathscr{P}_{\text{full}}\cap\mathscr{A}^k)$.
2. *The controller* $\mathfrak{C}$ *implements* $\mathscr{K}$ *by partial interconnection w.r.t.* $\mathscr{P}_{\text{full}}$ *if and only if the controller* $\mathfrak{C}+(\mathscr{P}_{\text{full}}\cap\mathscr{A}^k)$ *implements* $\mathfrak{C}_{\text{can}}(\mathscr{K})$ *by full interconnection w.r.t.* $\mathrm{pr}_2(\mathscr{P}_{\text{full}})$.
3. *The controller* $\mathfrak{C}$ *regularly implements* $\mathscr{K}$ *by partial interconnection w.r.t.* $\mathscr{P}_{\text{full}}$ *if and only if the controller* $\mathfrak{C}+(\mathscr{P}_{\text{full}}\cap\mathscr{A}^k)$ *regularly implements* $\mathfrak{C}_{\text{can}}(\mathscr{K})$ *by full interconnection w.r.t.* $\mathrm{pr}_2(\mathscr{P}_{\text{full}})$.
4. *If the controller* $\mathfrak{C}$ *implements the canonical controller by full interconnection w.r.t.* $\mathrm{pr}_2(\mathscr{P}_{\text{full}})$, *then it implements* $\mathscr{K}$ *by partial interconnection w.r.t.* $\mathscr{P}_{\text{full}}$.
5. *If the controller* $\mathfrak{C}$ *regularly implements the canonical controller by full interconnection w.r.t.* $\mathrm{pr}_2(\mathscr{P}_{\text{full}})$, *then it regularly implements* $\mathscr{K}$ *by partial interconnection w.r.t.* $\mathscr{P}_{\text{full}}$.

From this, Proposition 3 also follows as an immediate corollary.

We now study the following question. Suppose we have a linear nD behavior $\mathscr{K}$ such that $\mathscr{P}_{\text{full}}\cap\mathscr{A}^q\subseteq\mathscr{K}\subseteq\mathrm{pr}_1(\mathscr{P}_{\text{full}})$. Suppose $\mathscr{K}$ is regularly implementable by partial interconnection. Is it then true that every $\mathscr{K}'$ between $\mathscr{K}$ and $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ is also regularly implementable? For the case $n=1$ this is indeed true: in that case $\mathscr{K}$ is regularly implementable if and only if $\mathscr{K}+\mathrm{pr}_1(\mathscr{P}_{\text{full}})_{\text{cont}}=\mathrm{pr}_1(\mathscr{P}_{\text{full}})$, where $\mathrm{pr}_1(\mathscr{P}_{\text{full}})_{\text{cont}}$ is the controllable part of $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ (see [2, Theorem 4]). Obviously, any $\mathscr{K}'$ between $\mathscr{K}$ and $\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ then also satisfies $\mathscr{K}'+\mathrm{pr}_1(\mathscr{P}_{\text{full}})_{\text{cont}}=\mathrm{pr}_1(\mathscr{P}_{\text{full}})$, so is regularly implementable. It turns out that also for $n\geqslant 2$ the answer to the question is affirmative. This follows from the following lemma which states that if there is a 'good' $\mathscr{C}$ for the module $\mathscr{N}$, then there is a 'good' $\mathscr{C}$ for every module $\mathscr{N}'$ between $\mathscr{M}\cap\mathscr{D}^k$ and $\mathscr{N}$:

Theorem 15. *Let* $\mathscr{M}\subseteq\mathscr{D}^q\oplus\mathscr{D}^k$ *and* $\mathscr{N}\subseteq\mathscr{D}^q$ *be submodules such that* $\mathscr{M}\cap\mathscr{D}^q\subseteq\mathscr{N}\subseteq\mathrm{pr}_1(\mathscr{M})$. *Assume there exists a module* $\mathscr{C}\subseteq\mathscr{D}^k$ *such that* $(\mathscr{M}+\mathscr{C})\cap\mathscr{D}^q=\mathscr{N}$ *and* $\mathscr{M}\cap\mathscr{C}=\{0\}$. *Then for every module* $\mathscr{N}'\subseteq\mathscr{D}^q$ *such that* $\mathscr{M}\cap\mathscr{D}^q\subseteq\mathscr{N}'\subseteq\mathscr{N}$ *there exists* $\mathscr{C}'\subseteq\mathscr{D}^k$ *such that* $(\mathscr{M}+\mathscr{C}')\cap\mathscr{D}^q=\mathscr{N}'$ *and* $\mathscr{M}\cap\mathscr{C}'=\{0\}$.

Proof. It follows from Lemma 13, item 3, that $(\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M}))+(\mathscr{M}\cap\mathscr{D}^k)=\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. Let $\mathscr{N}'$ be any module between $\mathscr{M}\cap\mathscr{D}^q$ and $\mathscr{N}$. We claim that $(\mathscr{C}\cap\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}')))\oplus(\mathscr{M}\cap\mathscr{D}^k)=\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}'))$. The inclusion '$\subseteq$' is obvious. For the converse, let $(0,m_2)\in\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}'))$. There exists m_1 such that $(m_1,0)\in\mathscr{N}'$ and $(m_1,m_2)\in\mathscr{M}$. Since $\mathscr{N}'\subseteq\mathscr{N}$, we also have $(m_1,m_2)\in\tilde{\mathscr{M}}(\mathscr{N})$. Now, we can write $(0,m_2)=(0,c)+(0,m_2')$, with $(0,c)\in\mathscr{C}\cap\mathrm{pr}_2(\mathscr{M})$ and $(0,m_2')\in\mathscr{M}\cap\mathscr{D}^k$. This implies $(m_1,m_2)=(m_1,c)+(0,m_2')$ from which we deduce that $(m_1,c)\in\mathscr{M}$. Since $(m_1,0)\in\mathscr{N}$,we find that $(m_1,c)\in\tilde{\mathscr{M}}(\mathscr{N})$, so $(0,c)\in\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$. We conclude that $(0,m_2)\in(\mathrm{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))\cap\mathscr{C})+(\mathscr{M}\cap\mathscr{D}^k)$. $\square$

In terms of behaviors the previous theorem yields the following:

Corollary 16. *Let* $\mathscr{P}_{\text{full}}\subseteq\mathscr{A}^q\oplus\mathscr{A}^k$ *and* $\mathscr{K}\subseteq\mathscr{A}^q$ *be linear nD behaviors such that* $\mathscr{P}_{\text{full}}\cap\mathscr{A}^q\subseteq\mathscr{K}\subseteq\mathrm{pr}_1(\mathscr{P}_{\text{full}})$. *If* $\mathscr{K}$ *is regularly implementable by partial interconnection, then every linear nD behavior* $\mathscr{K}'$ *such that* $\mathscr{K}\subseteq\mathscr{K}'\subseteq\mathrm{pr}_1(\mathscr{P}_{\text{full}})$ *is regularly implementable by partial interconnection. In particular, if* $\mathscr{A}^q\cap\mathscr{P}_{\text{full}}$ *is regularly implementable by partial interconnection, then every implementable* $\mathscr{K}$ *is regularly implementable.*

To conclude this section, we will study the connection between the conditions of Proposition 2 and those of Proposition

3. In particular, we would like to understand how, for $n = 1$, the conditions of Proposition 2 follow from that of Proposition 3 and vice versa.

Let $\mathscr{P}_{\text{full}} \subseteq \mathscr{A}^q \oplus \mathscr{A}^k$ and $\mathscr{K} \subseteq \mathscr{A}^q$ be linear 1D systems such that $\mathscr{P}_{\text{full}} \cap \mathscr{A}^q \subseteq \mathscr{K} \subseteq \text{pr}_1(\mathscr{P}_{\text{full}})$. Let $\mathscr{M} \subseteq \mathscr{D}^q \oplus \mathscr{D}^k$ and $\mathscr{N} \subseteq \mathscr{D}^q$ be the corresponding modules. We have $\mathscr{M} \cap \mathscr{D}^q \subseteq \mathscr{N} \subseteq \text{pr}_1(\mathscr{M})$. The connection between the respective conditions 2 in Propositions 2 and 3 is given by the following:

Lemma 17. Assume that $\mathscr{N} \subseteq \text{pr}_1(\mathscr{M})$. Then $\text{pr}_2(\tilde{\mathscr{M}}\ (\mathscr{N}))/(\mathscr{M} \cap \mathscr{D}^k)$ is torsion-free if and only if $\mathscr{N}/(\mathscr{M} \cap \mathscr{D}^q)$ is torsion-free.

Proof. Let $[(0, m_2)] \neq 0$ be a torsion element of $\text{pr}_2(\tilde{\mathscr{M}}\ (\mathscr{N}))/(\mathscr{M} \cap \mathscr{D}^k)$, $(0, m_2) \in \text{pr}_2(\tilde{\mathscr{M}}\ (\mathscr{N}))$, $(0, m_2) \neq \mathscr{M} \cap \mathscr{D}^k$. There exists m_1 such that $(m_1, m_2) \in \mathscr{M}$ and $(m_1, 0) \in \mathscr{N}$. Clearly, $(m_1, 0) \neq \mathscr{M} \cap \mathscr{D}^q$ for otherwise $(0, m_2) \in \mathscr{M} \cap \mathscr{D}^k$. There exists $d \in \mathscr{D}$, $d \neq 0$, such that $(0, dm_2) \in \mathscr{M} \cap \mathscr{D}^k$. Also, $(dm_1, dm_2) \in \mathscr{M}$ so $(dm_1, 0) \in \mathscr{M} \cap \mathscr{D}^q$. This implies that $[(m_1, 0)] \neq 0$ is a torsion element of $\mathscr{N}/(\mathscr{M} \cap \mathscr{D}^q)$.

Conversely, let $[(m_1, 0)] \neq 0$ be a torsion element of $\mathscr{N}/(\mathscr{M} \cap \mathscr{D}^q)$, $(m_1, 0) \in \mathscr{N}$, $(m_1, 0) \neq \mathscr{M} \cap \mathscr{D}^q$. There exists m_2 such that $(m_1, m_2) \in \mathscr{M}$, so $(m_1, m_2) \in \tilde{\mathscr{M}}\ (\mathscr{N})$ and $(0, m_2) \in \text{pr}_2(\tilde{\mathscr{M}}\ (\mathscr{N}))$. Note that $(0, m_2) \neq \mathscr{M} \cap \mathscr{D}^k$ for otherwise $(m_1, 0) \in \mathscr{M} \cap \mathscr{D}^q$. There exists $d \in \mathscr{D}$, $d \neq 0$, such that $(dm_1, 0) \in \mathscr{M} \cap \mathscr{D}^q$. Also, $(dm_1, dm_2) \in \mathscr{M}$ so $(0, dm_2) \in \mathscr{M} \cap \mathscr{D}^k$. Hence, $[(0, m_2)] \neq 0$ is a torsion element of $\text{pr}_2(\tilde{\mathscr{M}}\ (\mathscr{N}))/(\mathscr{M} \cap \mathscr{D}^k)$. $\square$

For the case $n = 1$, if $\mathscr{M}_1$ and $\mathscr{M}_2$ are submodules of $\mathscr{D}^q$ with $\mathscr{M}_1 \subseteq \mathscr{M}_2$, then $\mathscr{M}_1$ is a direct summand of $\mathscr{M}_2$ if and only if $\mathscr{M}_2/\mathscr{M}_1$ is torsion-free. This can be proven using the fact that for $n = 1$, $\mathscr{D}$ is a principal ideal domain. Thus $\text{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))/(\mathscr{M} \cap \mathscr{D}^k)$ is torsion-free if and only if $\mathscr{M} \cap \mathscr{D}^k$ is a direct summand of $\text{pr}_2(\tilde{\mathscr{M}}(\mathscr{N}))$, equivalently, the canonical controller $\mathfrak{C}_{\text{can}}(\mathscr{K})$ is regularly implementable by full interconnection w.r.t. $\text{pr}_2(\mathscr{P}_{\text{full}})$. On the other hand, $\mathscr{N}/(\mathscr{M} \cap \mathscr{D}^q)$ is torsion-free if and only if $\mathscr{M} \cap \mathscr{D}^q$ is a direct summand of $\mathscr{N}$, equivalently, $\mathscr{K}$ is regularly implementable by full interconnection w.r.t. $\text{pr}_1(\mathscr{P}_{\text{full}})$. This gives a direct relation between the conditions 2 of Propositions 2 and 3.

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